Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part I

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Abstract

Let G be a finite group. An ordinary character of G is the character of a representation of G over a field of characteristic 0. In the p-modular representation theory of G,where p is a prime dividing the order of G, the ordinary irreducible characters of G are divided into disjoint sets called p-blocks which reflect the decomposition of the group algebra of G over a field of characteristic p into indecomposable two-sided ideals. An important problem is to classify the p-blocks, and a first step is to count the number of ordinary characters in a block.

The aim of Dade's Ordinary Conjecture (DOC) is to prove an alternating sum of the form

$$\sum_{C/G}(-1)^{|C|}k(N_G(C),B,d)=0, \qquad \forall d\geq 0$$

which counts the number of characters in B in terms of corresponding numbers in subgroups of G which are normalizers of chains of certain p-subgroups of G.

This has been shown for p-blocks, p dividing q, for $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$ and $\mathrm{U}_n(q)$. We prove DOC for $\mathrm{SU}_n(q)$. The main difficulties involved arise because the structure of the unitary groups is more complicated than that of the linear groups. In particular the cancellations in the alternating sum in the unitary case are very different from the cancellations that occur in the general linear case. A key result is that a version of Ku's parametrization of characters for $\mathrm{U}_n(q)$ survives restriction to $\mathrm{SU}_n(q)$.

This report is devoted firstly to some background and context for DOC for the finite special unitary groups. Then several reductions of the main alternating sum are completed resulting in an important reformulation of the main theorem. The alternating sum in this theorem is then immediately decomposed into two sub alternating sums. The final aim of this paper is to prove the first of these sub alternating sums, the so-called Levi sum.

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1 Introduction

Let G be a finite group. An ordinary representation of G is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

where V is a finite dimensional vector space over a field K of characteristic zero. Then V is the G-module, or KG-module, afforded by ρ. If we fix a basis for V, then $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}_n(K)$. Then the degree of the representation ρ is n. We define the character χ of ρ by $\chi(g) = \mathrm{Tr}(\rho(g))$. In general, though we may consider K to be the field of complex numbers, in practice we may take K to be a sufficiently large extension of $\mathbb{Q}$. A character χ is irreducible if the associated vector space V afforded by ρ has no proper nonzero KG-submodules. We will denote the ordinary irreducible characters of G by $\mathrm{Irr}(G)$. The set $\mathrm{Irr}(G)$ forms an orthonormal basis for the K-space of class functions on G.

Throughout this paper we assume that p is a fixed prime number. A modular representation of G is defined similarly, the key difference being that the module afforded is a vector space over a field of characteristic p. A modular representation of G is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(W)$$

where W is a vector space over a field k of characteristic p. Then W is the G-module, or kG-module, afforded by ρ. Restricting our attention to the elements of G with order prime to p, we may define a so called Brauer character, a complex valued class function on elements with order prime to p.

The philosophy of modular representations goes back to Brauer and in part relates "global information" to "p-local information". By global we mean the group G, and by p-local we mean subgroups related to p-subgroups of G, for example normalizers of p-subgroups. Information can refer to a variety of things including numbers of characters or character values.

Modular representations of G give rise to a partition of $\mathrm{Irr}(G)$ into blocks. Dade's Conjecture was first presented in a series of papers entitled *Counting Characters in Blocks*. The interplay between global and local blocks is informative in both directions. The conjecture involves an alternating sum involving characters of p-local subgroups. In order to state Dade's fairly elaborate conjecture we must first assemble some concepts.

1.1 Modular Representations

Let K be an algebraic number field, a splitting field for G and its subgroups. Let $\mathcal{O}$ be the ring of algebraic integers in K. Let $\mathcal{P}$ be a prime ideal in $\mathcal{O}$ containing p. Let $\mathcal{R}$ be the ring of $\mathcal{P}$-integral elements of K, i.e. the localization of $\mathcal{O}$ at $\mathcal{P}$. Let $\pi\mathcal{R}$ be the unique maximal ideal in $\mathcal{R}$ with respect to a valuation associated with the prime ideal $\mathcal{P}$. We define k to be the residue field $\mathcal{R}/\pi\mathcal{R}$ which is isomorphic to $\mathcal{O}/\mathcal{P}$. Then k has

characteristic p. The triple $(K, \mathcal{R}, k)$ is called a p-modular system. We may look at the following group algebras: KG, $\mathcal{R}G$, and kG.

We use $\mathcal{R}G$ in order to pass from ordinary to p-modular representations of G. As an $\mathcal{R}$-module, the $\mathcal{R}$-algebra $\mathcal{R}G$ is free and of finite rank. We have

$$KG = K \otimes_{\mathcal{R}} \mathcal{R}G \text{ and}$$
$$kG = k \otimes_{\mathcal{R}} \mathcal{R}G = {}^{\mathcal{R}G}/_{\pi\mathcal{R}G}\,.$$

A finitely generated $\mathcal{R}$-free $\mathcal{R}G$-module is called an $\mathcal{R}G$-lattice. If V is a KG-module then there exists an $\mathcal{R}G$-lattice M with

$$K \otimes_{\mathcal{R}} M \cong V.$$

Note that this is not unique as it depends on our choice of basis for V. Set

$$\bar{M} = k \otimes_{\mathcal{R}} M = M/\pi M \text{ as } kG\text{-module.}$$

Then $\bar{M}$ is a modular representation, or module, of G. The composition factors of $\bar{M}$ are unique up to isomorphism and do not depend on the choice of M.

The K-algebra KG is semi-simple and thus completely reducible. However kG is not semi-simple if $p \mid |G|$. Rather it can only be written as a sum of indecomposable two sided ideals. These indecomposable subalgebras of kG are called the p-blocks of kG. A decomposition of kG into blocks

$$kG = B_1 \oplus B_2 \oplus \cdots \oplus B_s$$

corresponds to a decomposition of the identity $1 = e_1 + e_2 + \cdots + e_s$ where the e_i are orthogonal primitive central idempotents in kG. This is given by $B_i = e_i kG$. The class sums of elements in G form a basis for both $Z(\mathcal{R}G)$ and $Z(kG)$. Hence reducing mod $\pi\mathcal{R}$ is a surjective map $Z(\mathcal{R}G) \longrightarrow Z(kG)$. We may lift the e_i to orthogonal primitive central idempotents f_i in $\mathcal{R}G$. Then the decomposition $1 = f_1 + f_2 + \cdots + f_s$ in $\mathcal{R}G$ corresponds to a decomposition of $\mathcal{R}G$ into two sided ideals also called the p-blocks of $\mathcal{R}G$.

$$\mathcal{R}G = \hat{B}_1 \oplus \hat{B}_2 \oplus \cdots \oplus \hat{B}_s$$

where $\hat{B}_i = f_i \mathcal{R}G$ and $B_i = k \otimes_{\mathcal{R}} \hat{B}_i$.

If V is an irreducible KG-modules then $f_i V = V$ for a unique f_i and $f_j V = 0$ for all $j \neq i$. We say that V belongs to the block $B = f_i KG$. If V affords the character χ, we also say that χ is in B. This gives rise to a partition of the set of ordinary characters of G into blocks. We may informally think of a p-block B of G as simultaneously being all of the following related objects:

- an indecomposable two-sided ideal kG-module $e_i kG$ for primitive idempotent $e_i \in Z(kG)$

- an indecomposable two-sided ideal $\mathcal{R}G$-module$f_i\mathcal{R}G$ for primitive idempotent $f_i \in Z(\mathcal{R}G)$ where f_i is the lift of e_i.
- the set of irreducible KG-modules V for which $f_iV = V$
- the set of ordinary characters of the KG-modules V as above

The two-sided ideal summands of kG (or $\mathcal{R}G$) are the same as the direct summands of kG (or $\mathcal{R}G$) as a $k(G \times G)$ (or $\mathcal{R}(G \times G)$) -module where the action of $k(G \times G)$ (or $\mathcal{R}(G \times G)$) on kG (or $\mathcal{R}G$) is given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.

Let B be a p-block of G. Then B has associated to it a p-group D called a defect group of B and a non-negative integer called the defect of B. The subgroup D is a minimal subgroup of G such that every B-module is a direct summand of an induced module from D ([18], p. 122). If D is such a defect group and $|D| = p^d$ then B has defect d. We define the defect of a character χ to be the maximum power of p dividing $\frac{|G|}{\chi(1)}$. Clearly, the defect of a character is inversely related to the power of the p-part of its degree. If B has defect d, then B contains a character of defect d and the defect of all other characters in B is less than or equal to d. We have two extremes. Write $|G| = p^e m$ where $p \nmid m$. If B contains a linear character then the defect of B is e. In this case we say B has full defect. For example the block containing the trivial module K, equivalently containing the trivial character, is called the principal block and has full defect. If B contains a character of degree divisible by p^e, then B has zero defect. It turns out that a block B of defect zero contains exactly one character ([6], Proposition 56.31).

Brauer's First Main Theorem states that if D is a p-subgroup of G then there exists a bijection between blocks B of G with defect group D and blocks b of $N_G(D)$ with defect group D. Let $H \leq G$ satisfy $DC_G(D) \leq H \leq N_G(D)$. Let B be a block of G and b be a block of H. We say b induces to B and write $b^G = B$ if b, as a $k(H \times H)$-module, is a summand in the restriction $B_{H\times H}$ of the $k(G \times G)$-module B to $H \times H$ and that B is the only block for which this holds ([1], p.). For H as above, b^G is always defined.

1.2 Statement of Dade's Ordinary Conjecture

Let G be a finite group and p a prime. Given a chain of p-subgroups $C : U_0 < U_1 < \cdots < U_l$ in G we define the length of C, $|C| = l$. We say that C is radical if $U_0 = O_p(G)$, the maximal normal p-subgroup of G and $U_i = O_p(\cap_{j=0}^{i} N_G(U_j))$ for $1 \leq i \leq l$. Let $N_G(C)$ denote $\cap_{j=0}^{l} N_G(U_j)$. Observe that if two chains C_1 and C_2 are conjugate to one another, then $N_G(C_1) \cong N_G(C_2)$. If b is a p-block of $N_G(C)$, then $b^G = B$ is defined. Let

$$\mathrm{Irr}(N_G(C), B, d) = \left\{\psi \in \mathrm{Irr}(N_G(C)) \mid \psi \in b \text{ where } b^G = B \text{ and } \psi \text{ has defect } d\right\}.$$

We will set $k(N_G(C), B, d) = |\mathrm{Irr}(N_G(C), B, d)|$.

Conjecture 1.1 ***Dade's Ordinary Conjecture (DOC)*** *([7], Conjecture 6.3). Let G be a finite group with $O_p(G) = 1$ so that all radical chains in G begin with the trivial group $U_0 = 1$. Let B be a block of G of nonzero defect. Then the following holds:*

$$\sum\nolimits'(-1)^{|C|}k(N_G(C), B, d) = 0, \qquad \forall d \geq 0$$

where $|C| = l$ is the length of C and $\sum'$ indicates the sum over a set of representatives of conjugacy classes of radical chains in G.

1.3 Refinement of DOC for Certain Finite Reductive Groups

DOC reduces nicely for certain finite reductive groups in the defining characteristic. Let G be a finite reductive group of characteristic p. Then G is the group of fixed points of a Frobenius endomorphism of a connected reductive algebraic group. We consider the p-blocks of G. Let I be an index set for the distinguished generators of the Weyl group W of G. Let B be a Borel subgroup of G. In this paper P_J will denote the parabolic subgroup $BW_{I\setminus J}B$. For example, $P_I = B$ is the Borel subgroup of G (rather than $P_\emptyset$). It is also useful to think of parabolics indexed in the following way: If $\{P_j \mid j \in I\}$ is a complete set of maximal parabolic subgroups in G, then

$$P_J = \bigcap_{j \in J} P_j.$$

Let $C : U_0 < U_1 < \cdots < U_l$ be a radical chain of p-subgroups in G. Then $U_0 = O_p(G) = 1$. Moreover, $U_1 = O_p(N_G(U_1))$ and hence, by ([3], Corollary), U_1 must be the unipotent radical U_J of a parabolic subgroup P_J of G with $N_G(U_J) = P_J$. We have the familiar Levi decomposition $P_J = L_J U_J$. It is obvious that $U_1 \subseteq B$. Notice that $P_J/U_J \cong L_J$ is itself a finite group of Lie type with Borel subgroup isomorphic to $B \cap L_J$. The quotient U_2/U_J is isomorphic to a p-group of $B \cap L_J$ and hence is isomorphic to a unipotent radical of a parabolic subgroup of L_J. Since $U_2 = O_p(P_J \cap N_G(U_2)) = O_p(N_{P_J}(U_2))$, we must have $U_2 = U_{J'}$ where $J' \supset J$. Hence C is a chain of unipotent radicals and $N_G(C)$ is equal to $N_G(U_l)$ the normalizer of the last term so that $N_G(C) = P_J$ for suitable J depending only on the last term of the chain C.

It turns out that there is considerable cancellation amongst the G-conjugacy classes of chains of unipotent radicals for G. The collection of all such chains C which terminate with a fixed U_J and thus have $N_G(C) = P_J$ cancels almost entirely, due to the alternating parity of the involved chains. One uncancelled chain remains of maximal length J. By a standard argument ([13], p.58),

$$\sum_C (-1)^{|C|}k(N_G(C), B, d) = \sum_{J \subseteq I} (-1)^{|J|}k(P_J, B, d)$$

where the sum on the left is taken over a set of representatives of G-conjugacy classes of chains of unipotent radicals.

The possible defect of p-blocks is well known for finite groups of Lie type, otherwise known as finite reductive groups, of characteristic p ([12]). The only possibilities are blocks of zero defect and blocks of full defect. In Humphreys' concluding remarks he notes that the number of blocks of zero defect is equal to the index of the derived subgroup G' in G and that the number of block of full defect is equal to the order of the center of G.

Let us now restrict our attention to $G = \mathrm{GL}_n(q)$, ${}_n(q)$, $\mathrm{U}_n(q)$, or $\mathrm{SU}_n(q)$, where q is a power of p. In fact we have a bijection from the set of p-blocks of P_J to the set of p-blocks of G of full defect. Indeed, the center of G is a torus and hence has order prime to p. Hence $O_{p'}(Z(G)) = Z(G)$ and $O_p(Z(G)) = 1$. Let $Z(G) = Z$. The group Z centralizes U_J so $U_J Z \subseteq U_J C_G(U_J)$ certainly holds. It happens that $U_J C_G(U_J) \subseteq U_J Z$ holds for these four families of groups. Thus by ([15], Lemma 2.1), ψ and ψ' lie in the same block b of P_J if and only if their restrictions to Z have the same constituent. In other words, P_J has $|Z|$ blocks and a block b of P_J is determined by a unique character $\rho \in \mathrm{Irr}(Z)$. The induced block $b^G = B$ is defined. B has full defect and is determined by the same ρ. The proof for $\mathrm{U}_n(q)$ is analogous to the proof for $\mathrm{GL}_n(q)$ in ([15], Lemma 2.1). If $\psi \in \mathrm{Irr}(P_J)$ restricted to Z contains ρ we will say that ψ lies over ρ.

Write $|G| = p^e m$. Each parabolic subgroup P_J contains U_I the unipotent radical of the Borel subgroup of G. This is a Sylow p-subgroup of G. Hence $|P_J|$ is divisible by p^e for every $J \subseteq I$. As noted above the defect of a character is inversely related to the power of p dividing its degree. If the p-part of $\psi(1)$ is p^a for $\psi \in \mathrm{Irr}(P_J)$, then the defect of ψ is $e - a$. Hence it is equivalent to count characters by their so called p-height rather than their defect.

Definition 1.2 *We define the p-height of ψ to be d if $p^d \| \psi(1)$. Similarly, we define the q-height ψ to be d if $q^d \| \psi(1)$.*

Remark: This definition is not entirely standard. In the literature p-height is generally defined with reference to the defect of the block containing the character. For example Brauer's definition of height in his Height Conjecture is more standard. However if ψ is in a block of full defect, then the p-height as it is usually defined is equal to the maximal power of p dividing $\psi(1)$ and hence our definition coincides with the standard.

Let ρ be an irreducible character of the center of G and define

$$k_d(P_J, \rho) = \left| \left\{ \psi \in \mathrm{Irr}(P_J) \mid \psi \text{ lies over } \rho \text{ and } p^d \| \psi(1) \right\} \right|.$$

Then DOC is equivalent to the following:

Conjecture 1.3 *Let $q = p^a$. Let $G = \mathrm{GL}_n(q)$, ${}_n(q)$, $\mathrm{U}_n(q)$, or $\mathrm{SU}_n(q)$, with parabolic subgroups P_J indexed by subsets $J \subseteq I$. Let Z be the center of G. Let $|G| = p^e m$ where*

$p \nmid m$. *Then* $\forall \rho \in \mathrm{Irr}(Z)$

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho) = 0 \ \forall d \ , \ 0 \leq d < e.$$

If G is either of $\mathrm{GL}_n(q)$ or $\mathrm{U}_n(q)$, then the p part of the degree of characters in $\mathrm{Irr}(G)$ are powers of q. Indeed, this is well known for $\mathrm{GL}_n(q)$ and follows for $\mathrm{U}_n(q)$ by replacing q in the $\mathrm{GL}_n(q)$-character theory by $-q$ in the $\mathrm{U}_n(q)$-theory, by Ennola's Conjecture, now proved. For $G = \mathrm{GL}_n(q)$, or $\mathrm{U}_n(q)$, let $S =_n (q)$, or $\mathrm{SU}_n(q)$ respectively. The group S is the kernel of the determinant map on G. Moreover the quotient G/S is cyclic of order $q-1$ or $q+1$ respectively, in either case prime to p. Take any $\psi \in \mathrm{Irr}(S)$, then there exists $\chi \in \mathrm{Irr}(G)$ such that ψ is a constituent of the restriction of χ to S. By Frobenius reciprocity, we may choose any irreducible χ appearing in the induction of ψ from S to G. Then, by Theorem 2.12 and Lemma 2.15 in the next section,

$$\chi|_S = \psi_1 + \psi_2 + \cdots + \psi_r \text{ where the } \psi_i \in \mathrm{Irr}(S) \text{ are } G\text{-conjugates of } \psi$$

and r divides $|G/S|$. Thus

$$\chi(1) = \psi_1(1) + \psi_2(1) + \cdots + \psi_r(1).$$

Since r is prime to p, it follows from Clifford theory that the p-height of ψ is equal to the p-height of χ and hence is also a power of q. Suppose $\chi \in \mathrm{Irr}(G)$, or $\mathrm{Irr}(S)$ has p-height d. Then d is certainly divisible by a so χ has q-height d/a. It turns out that from Olsson and Uno's construction for $\mathrm{GL}_n(q^2)$ and Ku's construction for $\mathrm{U}_n(q)$ the characters of parabolic subgroups of G also have degrees with p part equal to a power of q. As we will see in section 4, parabolic subgroups of S are in fact the kernel of the determinant map restricted to parabolic subgroups of G. Thus, by the same reasoning as above, they also have degrees with p part equal to a power of q. Hence in statement 1.3 of DOC, for d not divisible by a the left hand side of the sum is empty and so vacuously true. This allows us to simplify our notation by counting characters via q-height rather than p-height. Henceforth and for the rest of this thesis we redefine the subscript d so that it indicates q-height so for example $\mathrm{Irr}_d(P_J, \rho)$ will denote irreducible characters of P_J lying over ρ with q-height d.

1.4 Some Results for Dade's Conjecture and Implications

We summarize the cases for which some version of Dade's Conjecture has been shown, including the result of this paper (and its successor, *Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part II*). References for this section are ([11], Section 5) and on the web at ([16]).

1. Classical Groups:

$\mathrm{GL}_n(q)$	ord., $p \mid q$	Olssen, Uno
$\mathrm{U}_n(q)$	ord., $p \mid q$	Ku
$\mathrm{GL}_n(q)$, $\mathrm{U}_n(q)$	invar., $p \nmid q$	An
$Sp_{2n}(q)$, $SO^{\pm}_m(q)$	ord., $p \nmid q$, p, q odd	An
L_2	final	Dade
L_3	final, $p \mid q$	Dade
L_n	ord., $p \mid q$	Sukizaki
$Sp_4(2^n)$	final, $p = 2$	An, Himstedt, Huang
$SU_4(2^{2n})$		
$Sp_4(q)$	invar., $p \mid q$, p odd	An, Himstedt, Huang, Yamada
$\mathrm{SU}_n(q)$	ord., $p \mid q$	Bird

2. Sporadic Simple Groups:

M_{11}, M_{12}, J_1, J_2	final	Dade
M_{22}	final	Huang
M_{23}, M_{24}	final	Schwartz, An, Conder
J_3	final	Kotlica
McL	final	Murray, Entz, Pahlings
Ru	final	Dade, An, O'Brien
He	final	An
HS	final	Hassan, Horváth
Co_1	final	An, O'Brien
Co_2	final	An, O'Brien
Co_3	final	An
Suz	final	Himstedt
$O'N$	final	An, O'Brien, Uno, Yoshiara
Th	final	Uno
Ly	final	Sawabe, Uno
HN	final	An, O'Brien
Fi_{23}	final	An, O'Brien
Fi_{22}	invar.	An, O'Brien
J_4		An, O'Brien, Wilson
B	p odd	An, Wilson
${Fi'}_{24}$		An, Cannon, O'Brien, Unger

3. Exceptional Groups:

${}^2B_2(2^{2n+1})$	final	Dade
${}^2G_2(3^{2n+1})$	final	$p \neq 3$ An, $p = 3$ Eaton
$G_2(q)$	final, 2, $3 \mid q$, $p \nmid q$ $q \neq 3, 4$	An
$G_2(q)$	final, $p \mid q (p \geq 5)$, $q = 3, 4$	Huang
${}^3D_4(q)$	final, $p \nmid q$	An
${}^3D_4(q)$	final, $p \mid q$ ($p = 2$ or odd)	An, Himstedt, Huang
${}^2F_4(2^{2n+1})$	ord., $p \neq 2$	An
${}^2F_4(2^{2n+1})$	final, $p = 2$	Himstedt, Huang
${}^2F_4(2)'$	final	An

4. Other cases:

S_n	ord., $p \neq 2$	Olssen, Uno
S_n	ord., $p = 2$	An
A_n, abelian defect	ord.	Fong, Harris
Cyclic defect group	final	Dade
Tame block	invar.	Uno
Abelian defect unipotent blocks	ord.	Broué, Malle, Michel
Abelian defect principal blocks	ord., $p = 2$	Fong, Harris
Abelian defect some cases	ord.	Piug, Usami
p-solvable	proj.	Robinson
$O_p(G)$ cyclic, $O_p(G)/G$ p-Sylow TI	proj.	Eaton
Nilpotent blocks		

The sequence of most interest with respect to this paper is the following: Olsson and Uno proved DOC for $\mathrm{GL}_n(q)$ in the defining characteristic [15]. Sukizaki proved it for ${}_n(q)$ also in the defining characteristic [21]. Chao Ku verified DOC in his doctoral thesis for $\mathrm{U}_n(q)$.

Assuming that Dade's Ordinary Conjecture is true for all finite groups implies a number of other conjectures. In this sense DOC encodes a variety of information. DOC implies Alperin's Weight Conjecture which counts Brauer characters. DOC implies the Alperin-McKay Conjecture which is a refinement of the McKay Conjecture. DOC also implies one direction of Brauer's Height Conjecture which involves abelian defect groups.

2 Preliminaries

We begin these preliminaries with our definition of the finite unitary and special unitary groups. Throughout this article q is a fixed power of the prime p. Let $K = \overline{F_q}$, and $\widetilde{G} = \mathrm{GL}_n(K)$. Define the matrix

$$M = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

Define the following Frobenius map F on $\widetilde{G}$ by:

$$F(a_{i,j}) = M(a_{j,i}^q)^{-1}M^{-1}.$$

The group of fixed points $\widetilde{G}^F$ is the finite unitary group $\mathrm{U}_n(q)$, i.e.

$$\mathrm{U}_n(q) = \{(a_{i,j}) \mid M = (a_{i,j})M(a_{j,i}^q)\}.$$

Clearly $\mathrm{U}_n(q) \leq \mathrm{GL}_n(q^2)$. The advantage of this definition is that F fixes the subgroup of upper triangular matrices in $\mathrm{GL}_n(q^2)$. We can define the special unitary groups in two equivalent ways. On the one hand, the group of fixed points of ${}_n(K)$ under F is $\mathrm{SU}_n(q)$. On the other hand,

$$\mathrm{SU}_n(q) = \{A \in \mathrm{U}_n(q) \mid \det(A) = 1\}.$$

The Weyl group W of $\mathrm{U}_n(q)$ is of type B_m, where $n = 2m$, or $2m+1$, and is isomorphic to the wreath product $C_2 \wr S_m$. The symmetric group on m elements is generated by reflections indexed by $\{1, 2, \dots, m-1\}$ and the cyclic group of order 2 is generated by the reflection indexed by $\{m\}$. With this identification, the distinguished generators of W may be indexed by $I = \{1, 2, \dots, m\}$ denoted by $[m]$.

2.1 Some Notation

Throughout this article we will make use of the following notation. Let q be a fixed power of prime p. We consider the finite field F_{q^2} and its group of units $F_{q^2}^*$. For divisors h of $q^2 - 1$, let $\mathbb{C}_h$ denote the cyclic subgroup of order h in $F_{q^2}^*$. So in particular $\mathbb{C}_{q+1}$ denotes the cyclic subgroup of order $q+1$ in $\mathbb{C}_{q^2-1}$.

2.2 On radical p-chains

In order to reformulate DOC for the finite special unitary groups we will need the following proposition due to Sukizaki.

Proposition 2.1 *([21], Proposition 2.1) Let G be a finite group and let H be a subgroup of G. If H contains all p-subgroups of G and satisfies $O_p(G) = O_p(H)$, then any radical p-chain of H is a radical p-chain of G.*

2.3 Certain functions on partitions

In this section we are following the development of Olsson and Uno ([15]), Sukizaki ([21]), and Ku ([14]). To that end we discuss partitions. Further we define two important functions α and β on pairs (μ, a) where μ is a partition and a is a field element. The function α was defined in ([15], p.363). The function β was introduced as a unitary version of α and was defined in ([14], p.16). These functions are involved in expressing the q-height of characters. Further, it turns out that they are also involved in the splitting of characters upon restriction to certain subgroups. We assert some combinatorial facts about the behavior of these functions.

Let $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$, where $a_1 > a_2 \cdots > a_r > 0$. We define $|\mu| = \sum_{i=1}^{r} l_i a_i = n$. Let the number of distinct parts of the partition be $\delta(\mu) = r$ and the length of the partition $l(\mu) = \sum_{i=1}^{r} l_i$. We define $\gamma(\mu) = \gcd(a_1, a_2, \ldots, a_r)$ and $\lambda(\mu) = \gcd(l_1, l_2, \ldots, l_r)$.

Given $\mu_1 \vdash n_1$ and $\mu_2 \vdash n_2$ we can define $2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2$. In order to define this new partition write $\mu_i = (1^{m_{i1}}, 2^{m_{i2}}, \ldots, n_i^{m_{in_i}})$, so that for nonzero m_{it}, the integer t appears in μ_i with multiplicity m_{it}. Then define $2\mu_1 \cup \mu_2 = (1^{2m_{11}+m_{21}}, 2^{2m_{12}+m_{22}}, \ldots, n^{2m_{1n}+m_{2n}})$.

Definition 2.2 *Let $a \in \mathbb{C}_{q^2-1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. We define a function α by*

$$\alpha(\mu, a) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q^2-1})^r \mid (-1)^n {x_1}^{a_1} x_2^{a_2} \ldots x_r^{a_r} = a\}|.$$

Lemma 2.3 *With μ defined as above and $a \in \mathbb{C}_{q^2-1}$ we have*

$$\alpha(\mu, a) = (q^2 - 1)^{r-1} \alpha(\gamma, a)$$

where $\gamma = \gamma(\mu)$ and

$$\alpha(\gamma, a) = \begin{cases} gcd(q^2 - 1, \gamma), & \text{if } a \in \mathbb{C}_{(q^2-1)/gcd(q^2-1,\gamma)}; \\ 0, & \text{otherwise.} \end{cases}$$

See ([15], p.363) for the proof.

Definition 2.4 *Let $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. We define a function β*

$$\beta(\mu, b) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q+1})^r \mid (-1)^n {x_1}^{a_1} x_2^{a_2} \ldots x_r^{a_r} = b\}|.$$

Lemma 2.5 *With μ defined as above and $b \in \mathbb{C}_{q+1}$ we have*

$$\beta(\mu, b) = (q + 1)^{r-1} \beta(\gamma, b)$$

where $\gamma = \gamma(\mu)$ and

$$\beta(\gamma, b) = \begin{cases} gcd(q + 1, \gamma), & \text{if } b \in \mathbb{C}_{(q+1)/gcd(q+1,\gamma)}; \\ 0, & \text{otherwise.} \end{cases}$$

The proof is similar to the proof of Lemma 2.3.

In practice we will be restricting our attention to elements b in $\mathbb{C}_{q+1}$ and hence have need of the following modification of our α function which was defined in ([14], p.16).

Definition 2.6 *For $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$ we define*

$$\overline{\beta}(\mu, b) = \sum_{\substack{a \in \mathbb{C}_{q^2-1} \\ a^{q-1}=b}} \alpha(\mu, a).$$

Some important technical facts from ([14], [15], and [21]) regarding α, β, and $\overline{\beta}$ are summarized in the following lemmas.

Lemma 2.7 *If $(k) \vdash k$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$, then*

$$\sum_{\substack{b_1, b_2 \in \mathbb{C}_{q+1} \\ b_1 b_2 = b}} \beta((k), b_1)\beta(\mu, b_2) = \beta(\lambda, b)$$

where $\lambda = ((a_1 + k)^{l_1}, (a_2 + k)^{l_2}, \ldots, (a_r + k)^{l_r}, k^x) \vdash (n + (l(\mu) + x)k)$.

Lemma 2.8 *If $\mu_i \vdash n_i$, for $i = 1, 2$, and $\mu = 2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2$, then*

$$\sum_{\substack{b_1, b_2 \in \mathbb{C}_{q+1} \\ b_1 b_2 = b}} \overline{\beta}(\mu_1, b_1)\beta(\mu_2, b_2) = (q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1, \mu_2)}\beta(\mu, b)$$

where $c(\mu_1, \mu_2)$ is the number of distinct entries that μ_1 and μ_2 have in common.

Lemma 2.9 *If $\mu \vdash n$, then*

$$\sum_{\substack{(\mu_1, \mu_2) \\ \mu = 2\mu_1 \cup \mu_2}} q^{2(l(\mu_1) - \delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1, \mu_2)} = q^{l(\mu) - \delta(\mu)}.$$

Notice that this sum is taken over all pairs of partitions (μ_1, μ_2) such that $\mu = 2\mu_1 \cup \mu_2$.

These results are proved in ([14]). We mention that the last 2.9 is proved by associating to the pair (μ_1, μ_2) a matrix and its shadow which are defined as follows:

Definition 2.10 *For two partitions $\mu_i = (t^{m_{it}})$, $i = 1, 2$, with $\mu = (2\mu_1 \cup \mu_2) \vdash n$ we define the 2 by n matrix*

$$A(\mu_1, \mu_2) = A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix}$$

Given such an A we define the shadow of A to be the 2 by n matrix (c_{ij}) where the ij-entry

$$c_{ij} = \begin{cases} 1, & \text{if } m_{ij} \text{ is nonzero;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus taking the sum over pairs (μ_1, μ_2) in Lemma 2.9 is equivalent to taking the sum over possible matrices $A(\mu_1, \mu_2)$. We also remark that with this definition of the shadow, the number of entries that μ_1 and μ_2 have in common $c(\mu_1, \mu_2) = \sum_{t=1}^{n} c_{1t}c_{2t}$.

Suppose that we have a pair (μ_1, μ_2) with $\mu = 2\mu_1 \cup \mu_2$ a partition of n and all nonzero multiplicities of μ_i, $i = 1, 2$, are divisible by a fixed integer j, i.e. $j | \gcd(\lambda(\mu_1), \lambda(\mu_2))$. First of all it is clear that $j|\lambda(\mu)$. Write $\mu = (t^{m_t})$ so that $2m_{1t}+m_{2t} = m_t$. Observe that $tm_t \leq n$ must hold. In particular $(n/j)m_{n/j} \leq n$ implies that $m_{n/j}$ is the last possibly nonzero exponent in μ. In other words the matrix $A(\mu_1, \mu_2)$ must have zero entries to the right of the n/j-column. Furthermore, A may be decomposed:

$$A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \end{pmatrix} = jB.$$

If we remove the zero columns to the right of the n/j-column of B, we obtain $A(\kappa_1, \kappa_2)$, the 2 by n/j matrix associated to $\kappa_1 = (t^{k_{1t}})$ and $\kappa_2 = (t^{k_{2t}})$ for $1 \leq t \leq n/j$. If $\mu = (t^{2m_{1t}+m_{2t}})$, then $\kappa = (t^{2k_{1t}+k_{2t}}) = 2\kappa_1 \cup \kappa_2$, a partition of n/j. We have the following equalities:

$$l(\mu_1)/j = l(\kappa_1); \quad \delta(\mu_1) = \delta(\kappa_1);$$
$$c(\mu_1, \mu_2) = c(\kappa_1, \kappa_2); \quad l(\mu)/j = l(\kappa); \quad \delta(\mu) = \delta(\kappa).$$

Thus for a fixed partition $\mu \vdash n$

$$\sum_{\substack{(\mu_1,\mu_2) \\ \mu=2\mu_1\cup\mu_2 \\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}$$
$$= \sum_{\substack{(\kappa_1,\kappa_2) \\ \kappa=2\kappa_1\cup\kappa_2}} q^{2(l(\kappa_1)-\delta(\kappa_1))}(q-1)^{\delta(\kappa_1)}(q+1)^{c(\kappa_1,\kappa_2)}$$

Hence we have the following important corollary to 2.9:

Corollary 2.11 *If $\mu \vdash n$, then*

$$\sum_{\substack{(\mu_1,\mu_2) \\ \mu=2\mu_1\cup\mu_2 \\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = q^{l(\mu)/j-\delta(\mu)}.$$

2.4 Applications of the Clifford Theory

We will make abundant use of Clifford Theory. A reference for this section is ([6], Chapter 11). We summarize here the results that we need. In this section we assume that G is a finite group with a normal subgroup H. For $\psi \in \mathrm{Irr}(H)$ and $g \in G$ we define

the character ${}^g\psi$ by ${}^g\psi(h) = \psi(g^{-1}hg)$ for all $h \in H$. Let $T_G(\psi)$ denote the stabilizer of ψ in G so that

$$T_G(\psi) = \{g \in G \mid {}^g\psi = \psi\}.$$

We define a subset of characters in $\mathrm{Irr}(G)$

$$\mathrm{Irr}(G, \psi) = \{\chi \in \mathrm{Irr}(G) \mid (\chi|_H, \psi)_H \neq 0\}.$$

If $\chi \in \mathrm{Irr}(G, \psi)$ we will say that χ corresponds to ψ.

We define a subset of characters in $\mathrm{Irr}(H)$

$$\mathrm{Irr}(H, \chi) = \{\psi \in \mathrm{Irr}(H) \mid (\chi|_H, \psi)_H \neq 0\}.$$

This definition is equivalent to saying that the χ appear in the induced character of ψ to G. Thus we will say that ψ corresponds to χ if $\psi \in \mathrm{Irr}(H, \chi)$.

Theorem 2.12 *([6], Proposition 11.4) Let $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G, \psi)$. Then*

$$\chi|_H = e\left(\sum_{x \in G/T_G(\psi)} {}^x\psi\right)$$

where e is a positive integer.

Theorem 2.13 *([6], Theorem 11.5) Let $\psi \in \mathrm{Irr}(H)$ and suppose that $\psi = \tilde{\psi}|_H$ for some character $\tilde{\psi}$ of $T_G(\psi)$, that is, suppose that ψ can be extended to a character $\tilde{\psi}$ of $T_G(\psi)$. Write $T = T_G(\psi)$ then*

$$\mathrm{Irr}(T, \psi) = \{\ \theta\tilde{\psi} \mid \theta \in \mathrm{Irr}(T/H)\ \}, \quad \textit{and}$$
$$\mathrm{Irr}(G, \psi) = \{\ (\theta\tilde{\psi})^G \mid \theta \in \mathrm{Irr}(T/H)\ \}.$$

Here we regard θ as a character of T.

We have the following corollary which is a simple consequence of the transitivity of character induction.

Corollary 2.14 *If $\psi \in Irr(H)$ and P is any subgroup of G that contains $T_G(\psi)$ then*

$$|\mathrm{Irr}(P, \psi)| = |\mathrm{Irr}(G, \psi)|.$$

Moreover, there is a 1-1 correspondence between characters in each of these sets given by

$$\phi = (\theta\tilde{\psi})^P \leftrightarrow (\theta\tilde{\psi})^G = \chi.$$

If $\phi \in \mathrm{Irr}(P, \psi)$ corresponds to $\chi \in \mathrm{Irr}(G, \psi)$ and $\chi(1)$ has p-part equal to p^d then the p-part of $\phi(1)$ is $p^{d-d'}$ where $|P\backslash G|$ has p-part $p^{d'}$.

Lemma 2.15 *([22], Lemma 2.5) If G/H is cyclic then the following hold:*

1. *Characters of G restricted to H are multiplicity free. In other words $e = 1$ in 2.12.*

2. *Two characters of G either restrict to the same character of H or have disjoint irreducible components.*

3. *If $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G, \psi)$ then $|\mathrm{Irr}(H, \chi)| = |G|/(|H||\mathrm{Irr}(G, \psi)|)$.*

We have another important consequence of Clifford Theory concerning when an extension of a character exists when G is a semi-direct product.

Lemma 2.16 *([21], Theorem 2.5) Let G be a finite group with $G = P \ltimes M$.*

1. *If $\tau \in \mathrm{Irr}(M)$ is linear, then τ extends to an irreducible character $\tilde{\tau}$ of $T = T_G(\tau)$. Moreover*
$$\mathrm{Irr}(G, \tau) = \{(\theta\tilde{\tau})^G \mid \theta \in \mathrm{Irr}(T/M)\}.$$

2. *Let H be a normal subgroup of G containing M and suppose that G/H is cyclic. If $\theta \in \mathrm{Irr}(T/M)$, then*
$$|\mathrm{Irr}(H, (\theta\tilde{\tau})^G)| = |G : TH||\mathrm{Irr}(T_H(\tau), \theta)|.$$

We will be interested in the existence of extensions of non-linear characters of certain normal subgroups. The following result of Dade's on the extendibility of characters of normal extraspecial p-subgroups is certainly relevant.

Lemma 2.17 *([8]) Let E be an extra special p-group and $G = H \ltimes E$ with $Z(E) \leq Z(G)$. Assume that for each normal p'-subgroup K of H, the commutator subgroup $[K, E] = 1$. If $\psi \in \mathrm{Irr}(E)$ is non-linear, then ψ is extendible to G.*

2.5 On a Product of Groups

We will be examining the splitting of characters of direct products upon restriction to certain normal subgroups. We note that if $G = G_1 \times G_2$, then an irreducible character of G is of the form $\chi_1\chi_2$ where $\chi_i \in \mathrm{Irr}(G_i)$. We will have need of the following result.

Lemma 2.18 *([21]) Let $G = G_1 \times G_2$ where the group homomorphism $\phi_i : G_i \to F_{q^2}^*$ has image $\mathbb{C}_{h_i}$ for $i = 1, 2$. Set*
$$H = \{(g_1, g_2) \in G \mid \phi_1(g_1)\phi_2(g_2) = 1\}.$$
If χ_i has m_i irreducible constituents upon restriction to $\ker \phi_i$, then $\chi = \chi_1\chi_2$ restricted to H has m irreducible constituents, where
$$m = \frac{\gcd\left(m_1(q^2-1)/h_1, m_2(q^2-1)/h_2\right)}{\gcd\left((q^2-1)/h_1, (q^2-1)/h_2\right)}$$
.

2.6 Restriction of Characters to the Kernel of the Determinant Map

In this section we present results for $\mathrm{GL}_n(q^2)$ and $\mathrm{U}_n(q)$. Sukizaki's result is proved using G.I. Lehrer's work which uses an earlier parametrization of the characters of $\mathrm{GL}_n(q)$. We will use the more modern approach of Deligne-Lusztig theory. However, our goal remains the same in that we construct sequences of polynomials corresponding to characters and use them to count the number of characters.

Definition 2.19 *Let $G = \mathrm{GL}_n(q^2)$ or $\mathrm{U}_n(q)$. Given a homomorphism $\phi : G \to (F_{q^2})^*$ and $\rho \in \mathrm{Irr}(Z(G))$, we define the following:*

1. *Let $\mathrm{Irr}_d(G, \rho, \phi, j)$ be the set of irreducible ordinary characters χ of G with q-height d and lying over ρ such that the restriction of χ to the kernel of the map ϕ has j irreducible components.*

2. *Let $k_d(G, \rho, \phi, j)$ denote the number of irreducible ordinary characters χ of G with q-height d and lying over ρ such that the restriction of χ to the kernel of the map ϕ has j' irreducible components, where j divides j', i.e.*

$$k_d(G, \rho, \phi, j) = \sum_{\substack{j' \\ j|j'}} |\mathrm{Irr}_d(G, \rho, \phi, j')|.$$

We will be considering the determinant map on $\mathrm{U}_n(q)$ and certain subgroups. For a matrix element A, $\det(A)$ denotes the usual matrix determinant. We will consider subgroups of $\mathrm{U}_n(q)$ whose elements are block matrices. If A is a block matrix with block matrices $A_1, A_2, \ldots, A_s$ down its diagonal then $\det(A) = \det(A_1)\det(A_2)\cdots\det(A_s)$. Moreover, if certain of the A_i are repeated then $\det(A)$ may involve powers of $\det(A_i)$. Define $\det^h(A) = (\det(A))^h$. We apply definition 2.19 below with $\phi = \det$.

We now fix an isomorphism between $(\overline{F}_q)^*$ and $\mathrm{Irr}((\overline{F}_q)^*)$ and consider it fixed for the rest of this paper. In practice, we are primarily interested in the subgroup $F^*_{q^2}$. The group $Z = Z(\mathrm{GL}_n(q^2)) \cong F^*_{q^2} = \mathbb{C}_{q^2-1}$. Further, assume that the induced isomorphism of $\mathbb{C}_{q^2-1}$ with $\mathrm{Irr}(\mathbb{C}_{q^2-1})$ is given by the following. Let ε generate $\mathbb{C}_{q^2-1}$. Define the isomorphism via

$$\varepsilon \mapsto \rho_\varepsilon \text{ where } \rho_\varepsilon(\varepsilon) = e^{(2\pi i)/(q^2-1)}.$$

Under this isomorphism, $\rho \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$ corresponds to $a_\rho \in \mathbb{C}_{q^2-1}$. Equivalently $a \in \mathbb{C}_{q^2-1}$ corresponds to $\rho_a \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$. This induces an isomorphism of $\mathrm{Irr}(Z(\mathrm{U}_n(q)))$ with $\mathbb{C}_{q+1}$.

The following integer valued function on partitions of n is involved in the q-height of characters for $\mathrm{GL}_n(q^2)$ and $\mathrm{U}_n(q)$.

Definition 2.20 *We define* $n'(\mu)$*:*

$$n'(\mu) = \sum_{i=1}^{r} l_i \binom{a_i}{2}.$$

Our first proposition is a slight reformulation of Sukizaki's result in [21]. This is needed as our ground field is F_{q^2} rather than F_q. For $\mu \vdash n$, recall $\lambda(\mu)$, $l(\mu)$, and $\delta(\mu)$ defined on page 12.

Proposition 2.21 *([21], Lemma 4.1) Let* $\rho \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$*. Then*

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j \mid \gcd(q^2-1, \lambda(\mu))}} q^{2(l(\mu)/j - \delta(\mu))} \alpha(\mu, a_\rho).$$

For $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ we define

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det{}^{1-q}, j) = \sum_{\substack{\rho' \in \mathrm{Irr}(\mathbb{C}_{q^2-1}) \\ \rho'|_{\mathbb{C}_{q+1}} = \rho}} k_{2d}(\mathrm{GL}_n(q^2), \rho', det^{1-q}, j).$$

Then since $(q^2-1)/(q-1) = q+1$, by Sukizaki's equation following equation 3-5 in [21] we have a disjoint union

$$\mathrm{Irr}_{2d}(\mathrm{GL}_n(q^2), \rho', \det{}^{1-q}, j) = \bigsqcup_{\substack{j' \\ j = \gcd(q+1, j')}} \mathrm{Irr}_{2d}(\mathrm{GL}_n(q^2), \rho', \det, j').$$

This together with our definition of $\overline{\beta}$ from earlier in this section implies the following corollary:

Corollary 2.22 *Let* $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$*. Then*

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det{}^{1-q}, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j \mid \gcd(q+1, \lambda(\mu))}} q^{2(l(\mu)/j - \delta(\mu))} \overline{\beta}(\mu, a_\rho).$$

We present the case for $\mathrm{U}_n(q)$ now for completeness. We will prove this in the next section.

Proposition 2.23 *Let* $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$*. Then*

$$k_d(\mathrm{U}_n(q), \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j \mid \gcd(q+1, \lambda(\mu))}} q^{l(\mu)/j - \delta(\mu)} \beta(\mu, a_\rho)$$

Remark: Let χ be an irreducible character of $\mathrm{GL}_n(q^2)$. If $\chi\,|_{\ker\det}$ has j irreducible constituents, then j divides $\gcd(q^2-1,n)$. ([19], Theorem 4.7). If $\chi\,|_{\ker\det^{1-q}}$ has j irreducible constituents, then j divides $\gcd(q+1,n)$. Now let χ be an irreducible character of $\mathrm{U}_n(q)$. If $\chi\,|_{\ker\det}$ has j irreducible constituents, then j divides $\gcd(q+1,n)$. An identical theorem for the unitary case may be obtained making the following simple modifications to the proof of ([19], Theorem 4.7): Change Definition 4.6 by defining

$$M(d) = \{A \in \mathrm{U}_n(q) \mid \det A = \xi^{dk}\ , k = 1, \ldots, (q+1)/d\}$$

where $d = \gcd(n, q+1)$ and $\xi = \varepsilon^{1-q}$, a generator of the subgroup $\mathbb{C}_{q+1}$ in $F_{q^2}^*$. Then in Lemma 4.6 and Theorem 4.7 replace $\mathrm{GL}_n(q)$ with $\mathrm{U}_n(q)$ and ${}_n(q)$ with $\mathrm{SU}_n(q)$.

3 Characters of $\mathbf{U_n(q)}$ restricted to $\mathbf{SU_n(q)}$

In this section we prove Proposition 2.23. We start by parameterizing irreducible characters χ of $\mathrm{U}_n(q)$ via pairs (s, λ), and construct a unique sequence of polynomials $(h_1(x), h_2(x), \ldots)$ corresponding to χ. The subgroup $\mathrm{SU}_n(q)$ is normal in $\mathrm{U}_n(q)$ with cyclic quotient isomorphic to $\mathbb{C}_{q+1}$ which acts naturally on $\mathrm{Irr}(\mathrm{SU}_n(q))$ via $\mathrm{U}_n(q)$-conjugation. In the last section, we fixed an isomorphism $\mathbb{C}_{q^2-1} \cong \mathrm{Irr}(\mathbb{C}_{q^2-1})$ and hence we have an isomorphism $\mathbb{C}_{q+1} \simeq \mathrm{Irr}(\mathbb{C}_{q+1})$. The group $\mathrm{Irr}(\mathbb{C}_{q+1})$ acts on $\mathrm{Irr}(\mathrm{U}_n(q))$. Indeed, if $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ then we have a corresponding linear character of $\mathrm{U}_n(q)$ also denoted by ρ. Then $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ acts on $\mathrm{Irr}(\mathrm{U}_n(q))$ by

$$\chi \mapsto \rho \otimes \chi \text{ abbreviated by } \rho\chi.$$

Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$. By Clifford Theory, χ restricted to $\mathrm{SU}_n(q)$ is multiplicity free. If

$$\chi|_{\mathrm{SU}_n(q)} = \psi_1 + \psi_2 + \cdots \psi_j \qquad \text{where } \psi_i \in \mathrm{Irr}(\mathrm{SU}_n(q))$$

then the ψ_i are $\mathrm{U}_n(q)$-conjugates of one another.

The following lemma uses the well-known fact on characters of finite groups that if G is a finite group and $H \leq G$, and ϑ, η are characters of H, G respectively, then

$$\eta \mathrm{Ind}_H^G(\vartheta) = \mathrm{Ind}_H^G(\eta|_H \vartheta).$$

Lemma 3.1 *Let χ, $\chi' \in \mathrm{Irr}(\mathrm{U}_n(q))$. Then χ, χ' have the same restriction to $\mathrm{SU}_n(q)$ if and only if $\chi' = \rho\chi$ for some $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$.*

Proof: Let $\chi' = \rho\chi$. Since $\mathrm{SU}_n(q)$ is in the commutator subgroup of $\mathrm{U}_n(q)$, ρ is trivial on $\mathrm{SU}_n(q)$ and hence χ, χ' have the same restriction to $\mathrm{SU}_n(q)$.

Suppose ψ is a common constituent of χ, χ' restricted to $\mathrm{SU}_n(q)$. Let T be the stabilizer of ψ in $\mathrm{SU}_n(q)$. Then ψ extends to $\tilde{\psi} \in \mathrm{Irr}(T)$, and we have

$$\chi = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi}\phi), \qquad \chi' = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi}\phi')$$

where ϕ, ϕ' are lifts to T of characters of $T/\mathrm{SU}_n(q)$, denoted ϕ_1, ${\phi'}_1$. Then ϕ_1, ${\phi'}_1$ can be extended to characters ξ_1, ${\xi'}_1$ of $\mathrm{U}_n(q)/\mathrm{SU}_n(q)$, which can be lifted to characters ξ, ξ' of $\mathrm{U}_n(q)$. Then we have

$$\chi = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi})\xi, \qquad \chi' = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi})\xi'.$$

Thus χ, χ' differ by a linear character. Since every linear character of $\mathrm{U}_n(q)$ is of the form ρ_z for some $z \in \mathbb{C}_{q+1}$ we have $\chi' = \rho\chi$.

Let $\mathcal{E} = \{\psi_1, \psi_2, \ldots, \psi_j\}$ and let $\mathcal{F} = \{\chi_1, \chi_2, \ldots, \chi_r\}$ where the χ_i are the constituents of the induced character $\mathrm{Ind}_{\mathrm{SU}_n(q)}^{\mathrm{U}_n(q)}(\psi)$ for any $\psi \in \mathcal{E}$. Then $\mathcal{E}$ is a $\mathbb{C}_{q+1}$-stable subset of $\mathrm{Irr}(\mathrm{SU}_n(q))$ and $\mathcal{F}$ is a $\mathrm{Irr}(\mathbb{C}_{q+1})$-stable subset of $\mathrm{Irr}(\mathrm{U}_n(q))$. Hence $r = (q+1)/j$

and our original character χ is stabilized by $\rho_z \in \mathrm{Irr}(\mathbb{C}_{q+1})$ where $z \in \mathbb{C}_{q+1}$ is a primitive j-th root of unity, i.e.

$$\chi = \rho_z \chi.$$

This forces certain conditions on the coefficients in the polynomials in $(h_1(x), h_2(x), \ldots)$ corresponding to χ and allows us to count how many χ of a fixed q-height are fixed by a j-th root of unity.

3.1 Pairs (s, λ)

Let $K = \overline{F}_q$. Consider the algebraic group $\widetilde{G} = \mathrm{GL}_n(K)$ with Frobenius endomorphism defined in the last section, $F : \widetilde{G} \to \widetilde{G}$ by $F((a_{ij})) = M(a_{ji}^q)^{-1}M^{-1}$. Then let $G = \widetilde{G}^F = \mathrm{U}_n(q)$.

A reference for the following is ([10], section 1). A subgroup L of G is Levi if $L = \widetilde{L}^F$ for some F-stable Levi subgroup $\widetilde{L}$ of a parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$. For a Levi subgroup L of G, let R_L^G be the additive operator from $X(L)$ to $X(G)$ defined in the Deligne-Lusztig theory, where $X(L)$ and $X(G)$ are the character rings of representations of L and G over $\overline{\mathbb{Q}}_l$, an algebraic closure of the l-adic field $\overline{\mathbb{Q}}_l$ ($l \neq p$). Recall, in the previous section we fixed an isomorphism between $(\overline{F}_q)^*$ and $\mathrm{Irr}((\overline{F}_q)^*)$. Providing a coherent choice of roots of unity (via monomorphisms of multiplicative groups) has been made, this leads to an isomorphism

$$Z(L) \cong \mathrm{Irr}(Z(L)) = \mathrm{Hom}(Z(L), \overline{\mathbb{Q}}_l)$$

as in ([4], Section 8.2). Recall, ρ_s is the linear character of L corresponding to $s \in Z(L)$. We have a Jordan decomposition of characters of G. Namely the set of ordinary irreducible characters of G is in one-to-one correspondence with the set of pairs (s, λ). In our case, this means

$$\chi \leftrightarrow (s, \lambda)$$

where s is a representative of a semi-simple conjugacy class of G and λ is a unipotent character of $L = C_G(s)$, i.e. λ appears as a constituent of $R_T^L(1)$ for some maximal torus T of L.

Let $\epsilon_L = (-1)^d$ where d is the dimension of a maximal F_{q^2} split torus of L. Then

$$\chi = \epsilon_G \epsilon_L R_L^G(\rho_s \lambda) \text{ by ([10], p.116).}$$

Proposition 3.2 *For $\rho_z \in \mathrm{Irr}(Z(G))$, and $\chi \in \mathrm{Irr}(G)$*

$$\chi \leftrightarrow (s, \lambda) \Leftrightarrow \rho_z \chi \leftrightarrow (zs, \lambda).$$

where (s, λ) is the Jordan decomposition of χ.

Proof: Let $L = C_G(s) = C_G(sz)$. Then

$$\chi = \epsilon_G \epsilon_L R_L^G(\rho_s \lambda).$$

Moreover

$$\begin{aligned}\rho_z\chi &= \epsilon_G\epsilon_L R_L^G(\rho_z(\rho_s\lambda)), \text{ by ([4], Proposition 8.20)}\\ &= \epsilon_G\epsilon_L R_L^G((\rho_z\rho_s)\lambda),\end{aligned}$$

which corresponds to the pair (zs, λ).

Remark: The above discussion also holds for the finite group $\mathrm{GL}_n(q) = \widetilde{G}^{F'}$ where the Frobenius map is given by $F'((a_{i,j})) = (a_{i,j}^q)$.

3.2 Sequences of polynomials

In order to count efficiently we make use of polynomial sequences. We construct certain sequences which correspond to the irreducible characters of $\mathrm{U}_n(q)$. This identification arises naturally out of Deligne-Lusztig Theory. These polynomials encode information about both s and λ. This procedure is known in the case of $\mathrm{GL}_n(q)$ where if χ corresponds to the pair (s, λ) then χ corresponds to a sequence $(h_1(x), h_2(x), \ldots)$. The $h_i(x)$ are products of powers of irreducible polynomials over F_q which are elementary divisors of s; the powers of these divisors come from λ. In precisely the same spirit, irreducible characters of $\mathrm{U}_n(q)$ can be identified with sequences $(h_1(x), h_2(x), \ldots)$ where the $h_i(x)$ are products of powers of polynomials over F_{q^2}, appropriate for $\mathrm{U}_n(q)$, which are elementary divisors of s. We proceed with this identification.

It is well known that the conjugacy class of an element in $\mathrm{GL}_n(q^2)$ may be described by the elementary divisors of the rational canonical form. These divisors are powers of monic irreducible polynomials in $F_{q^2}[x]$ with non-zero roots. View $\mathrm{U}_n(q)$ as a subgroup of $\mathrm{GL}_n(q^2)$. Let $g \in \mathrm{GL}_n(q^2)$ have $\mathrm{GL}_n(q^2)$-conjugacy class $[g]$. The intersection $[g] \cap \mathrm{U}_n(q)$ is either a $\mathrm{U}_n(q)$-conjugacy class or is empty ([10], p.111). Let f be a monic irreducible polynomial in $F_{q^2}[x]$ of degree d with nonzero roots $\{\omega\}$. We define $\tilde{f}$ to be the polynomial in $F_{q^2}[x]$ with roots $\{\omega^{-q}\}$. Let $m_{f^i}(g)$ denote the multiplicity of f^i as an elementary divisor of g. Then $[g] \cap \mathrm{U}_n(q)$ is nonempty precisely when $m_{f^i}(g) = m_{\tilde{f}^i}(g)$ holds $\forall f$ and $\forall i$. Hence the conjugacy class of an element in $\mathrm{U}_n(q)$ is given by the elementary divisors of its rational canonical form and these divisors are powers of polynomials in a subset $\mathfrak{F}$ of $F_{q^2}[x]$.

Definition 3.3 *Let* $\mathfrak{F}_1 = \{f | f \neq x \text{ is monic, irreducible and } f = \tilde{f}\}$ *and let* $\mathfrak{F}_2 = \{f\tilde{f} | f \neq x \text{ is monic, irreducible and } f \neq \tilde{f}\}$. *Let* $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$.

Notice that for every polynomial $f \in \mathfrak{F}$, $f = \tilde{f}$. Members of $\mathfrak{F}_1$ have odd degree and members of $\mathfrak{F}_2$ have even degree. The latter fact is obvious. The former can be observed by noting that since f is irreducible the roots are the Galois conjugates of ω. Suppose that $d = 2k$. If $f = \tilde{f}$, then $\{\omega, \omega^{-q}, \ldots, \omega^{(-q)^{d-1}}\} = \{\omega^{-q}, \omega^{q^2}, \ldots, \omega^{(-q)^d}\}$. Hence $\omega = \omega^{(-q)^n} = \omega^{q^d}$ so $\omega \in F_{q^d}$. But F_{q^d} is an extension of F_{q^2} of degree k, hence f is reducible, a contradiction.

An element $g \in \mathrm{U}_n(q)$ is semi-simple if and only if $m_{f^i}(g) = 0$ for all $i > 0$. Given a semi-simple $s \in \mathrm{U}_n(q)$, we want to describe its centralizer $C_{\mathrm{U}_n(q)}(s)$. Let s have primary decomposition $s = \prod_{f \in \mathfrak{F}} s_f$ where s_f is the primary component corresponding to elementary divisor f. Let s have minimal polynomial $\min(x) = \prod_{f \in \mathfrak{F}} f$ and characteristic polynomial $ch(x) = \prod_{f \in \mathfrak{F}} f^{m_f(s)}$. Then s has rational canonical form

$$s = \bigoplus_{\mathfrak{F}} m_f(s)c(f)$$

where $c(f)$ denotes the $d_f \times d_f$ companion matrix of the polynomial f with degree d_f and for nonzero multiplicity $m_f(s)$, $m_f(s)c(f)$ denotes the $m_f(s)d_f \times m_f(s)d_f$ matrix with $m_f(s)$ copies of $c(f)$.

Proposition 3.4 *([10], Proposition 1A) Let s have primary decomposition $s = \prod_{f \in \mathfrak{F}} s_f$ and rational canonical form*

$$s = \bigoplus_{f \in \mathfrak{F}} m_f(s)c(f).$$

The structure of the centralizer of s is given by

$$C_{\mathrm{U}_n(q)}(s) = \prod_{f \in \mathfrak{F}} C(s_f), \text{ where}$$

1. *If $f \in \mathfrak{F}_1$, then $C(s_f) = \mathrm{U}_{m_f(s_f)}(F_f)$, where $|F_f : F_{q^2}| = \deg(f)$.*
2. *If $f \in \mathfrak{F}_2$, then $C(s_f) = \mathrm{GL}_{m_f(s_f)}(F_f)$, where $|F_f : F_{q^2}| = \frac{1}{2}\deg(f)$.*

Hence the centralizer of an element is a product of general linear and unitary groups. A unipotent character of such a product is a product of unipotent characters. Moreover, the unipotent characters of both the general linear and unitary groups are indexed by partitions of the dimension of the underlying vector space. In particular, the unipotent characters of $\mathrm{U}_n(q^m)$ and $\mathrm{GL}_n(q^m)$ are given by partitions of n, for any exponent m.

Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ correspond to the pair (s, λ). Since λ is a unipotent character of $C_{\mathrm{U}_n(q)}(s)$ it is a product of unipotent characters of the $C(s_f)$ which are general linear or unitary groups each of which corresponds to a partition $\mu_f \vdash m_f(s)$. Let $\mathfrak{P}$ denote the set of all partitions including the empty partition. We define the map

$$\begin{aligned} \Lambda : \mathfrak{F} &\longrightarrow \mathfrak{P} \\ f &\mapsto \mu_f. \end{aligned}$$

Notice that $\sum_{f \in \mathfrak{F}} d_f m_f(s) = n$.

Our construction is summarized in the following often quoted proposition which originates with Green's important paper on general linear characters, and has been modified for $\mathrm{U}_n(q)$ by several authors. Here we use the notation of Ku ([14]). For a partition μ recall the definitions of $|\mu|$ on page 12 and $n'(\mu)$ on page 18 in the previous section.

Proposition 3.5 *([14], Proposition 4.2.2 and Lemma 4.2.3) Let $\mathfrak{P}$ denote the set of all partitions of all integers $n > 0$, together with the empty partition. The irreducible characters of $\mathrm{U}_n(q)$ are in one-to-one correspondence with maps Λ from $\mathfrak{F}$ to $\mathfrak{P}$ which satisfy the following:*

$$\sum_{f \in \mathfrak{F}} |\Lambda(f)| d_f = n.$$

If $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ corresponds to such a map Λ, then the following hold:

1. *The q-height of χ is $\sum_{f \in \mathfrak{F}} d_f n'(\Lambda(f)')$ where $\Lambda(f)'$ is the conjugate partition of $\Lambda(f)$.*
2. *The character χ lies over $\rho \in \mathrm{Irr}(Z(\mathrm{U}_n(q)))$ where a_ρ is the product of the roots of $\prod_{f \in \mathfrak{F}} f^{|\Lambda(f)|}$.*

Let $\chi \in \mathrm{Irr}(G)$ be associated to the pair (s, λ) which is in turn associated to the map $\Lambda : \mathfrak{F} \to \mathfrak{P}$. For each $f \in \mathfrak{F}$, write the conjugate partition $\Lambda(f)' = (t^{m_{f,t}})$. Using these exponents, we may now define for χ a unique sequence of polynomials $(h_1(x), h_2(x), \ldots)$ by letting

$$h_i(x) = \prod_{f \in \mathfrak{F}} f^{m_{f,i}}.$$

We will be concerned with examining classes of irreducible characters which share certain properties. We want to group characters by their q-height and also by their splitting upon restriction to certain subgroups. To that end we make the following definition which will be of utmost importance in this endeavor.

Definition 3.6 *If χ determines the sequence $(h_1(x), h_2(x), \ldots)$, we will say that χ is of μ-type where $\mu = (t^{\deg(h_t(x))}) \vdash n$.*

3.3 Proof of Proposition 2.23

In this section we verify Proposition 2.23. Recall $k_d(\mathrm{U}_n(q), \rho, \det, j)$ is the number of $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ of q-height d lying over ρ such that $\chi|_{\mathrm{SU}_n(q)}$ has j' irreducible constituents where $j|j'$. Let $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. Let $\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)$ denote the irreducible characters of $\mathrm{U}_n(q)$ of μ-type lying over ρ. Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)$ correspond to (s, λ) and $(h_1, h_2, \ldots)$. Suppose ρ_z is the linear character of $\mathrm{U}_n(q)$ corresponding to $z \in \mathbb{C}_{q+1}$, a primitive j-th root of unity and that

$$\chi = \rho_z \chi.$$

By 3.2 $\rho_z \chi$ corresponds to (zs, λ). If h_i in $(h_1, h_2, \ldots)$ has roots $\{\omega\}$ then (zs, λ) corresponds to $(g_1, g_2, \ldots)$ where g_i has roots $\{z\omega\}$. Then we have

$$(h_1, h_2, \ldots) = (g_1, g_2, \ldots).$$

Since χ is of μ-type, $h_{a_i}(x)$ is a polynomial of degree l_i and hence $g_{a_i}(x)$ also has degree l_i. Let

$$\{\omega_{i,k} | 1 \leq k \leq l_i\} \text{ denote the roots of } h_{a_i}(x).$$

Then

$$\begin{aligned} h_{a_i}(x) &= \prod_{k=1}^{l_i}(x - \omega_{i,k}) & g_{a_i}(x) &= \prod_{k=1}^{l_i}(x - z\omega_{i,k}) \\ &= x^{l_i} + \cdots + b_{i,1}x + b_{i,0} & &= x^{l_i} + \cdots + z^{l_i-1}b_{i,1}x + z^{l_i}b_{i,0}. \end{aligned}$$

Recall $\lambda(\mu)$ was defined on page 12 and is equal to $\gcd(l_1, l_2, \ldots, l_r)$. Our first observation is that $b_{i,0}$ is nonzero. Hence $z^{l_i} = 1$ for each $i = 1, 2, \ldots, r$ thus j divides $\lambda(\mu)$. Secondly, we must have $b_{i,j}x^k = z^{l_i-k}b_{i,k}x^k$. If $l_i - k$ is not divisible by j, i.e. j doesn't divide k, the coefficient $b_{i,k} = 0$. This reduces the possible number of nonzero coefficients.

If χ lies over ρ then $(-1)^n \prod_{i=1}^r (b_{i,0})^{a_i} = a_\rho$ by construction. The $b_{i,k}$ are symmetric functions of the roots. Simplifying notation for a moment, since $h(x) = x^m + \ldots + b_1x + b_0$ is a product of polynomials in $\mathfrak{F}$, the coefficients satisfy $b_{m-i} = b_0 b_i^q$. If m is even $b_{m/2}^{1-q} = b_0$. Hence we have $(l_i/j - 1)/2$ degrees of freedom in the nonconstant coefficients, i.e. $q^{2(l_i/j-1)/2}$ choices for the $b_{i,k}$ and thus

$$|\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)| = q^{l(\mu)/j - \delta(\mu)}\beta(\mu, a_\rho).$$

The left hand side of the sum in Proposition 2.23 can now be evaluated.

$$\begin{aligned} k_d(\mathrm{U}_n(q), \rho, \det, j) &= \sum_{\substack{j' \\ j|j'}} |\mathrm{Irr}_d(\mathrm{U}_n(q), \rho, \det, j')| \\ &= \sum_{\substack{j' \\ j|j'}} \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j' | \gcd(q+1, \lambda(\mu))}} |\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)| \\ &= \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j | \gcd(q+1, \lambda(\mu))}} q^{l(\mu)/j - \delta(\mu)}\beta(\mu, a_\rho). \end{aligned}$$

4 The Finite Special Unitary Groups: A Reduction of DOC

In this section we make use of Clifford theory in the manner of Sukizaki ([21]) to reformulate DOC for the Special Unitary group $\mathrm{SU}_n(q)$ in terms of $\mathrm{U}_n(q)$. In this section we will distinguish between subgroups of $\mathrm{U}_n(q)$ and $\mathrm{SU}_n(q)$ with superscripts as indicated. Recall $I = [m]$ is our index set for the distinguished generators of the Weyl group for $\mathrm{U}_n(q)$. We have $n = 2m$ or $2m+1$. In keeping with established notation, let $K = \overline{F}_q$. Let $\widetilde{B}$ be the Borel subgroup of upper triangular matrices of the linear algebraic group $\widetilde{G} = \mathrm{GL}_n(K)$. A Frobenius endomorphism on $\widetilde{G}$ was defined by

$$F(a_{ij}) = M(a_{ji}^q)^{-1}M^{-1},$$

and the unitary group $\mathrm{U}_n(q) = \widetilde{G}^F$. The special unitary group is defined

$$\mathrm{SU}_n(q) = \{g \in \mathrm{U}_n(q) \mid \det g = 1\}.$$

Note that except for the cases $n = 2$ and $q \leq 3$, the derived subgroup $\widetilde{G}' =_n (K)$ so $\mathrm{SU}_n(q) = \widetilde{G}'^F$. The group of fixed points of $\widetilde{B}$ under F is a Borel subgroup for $\mathrm{U}_n(q)$. Let B^{U} be this subgroup. Notice that B^{U} is upper triangular. We will fix a Borel subgroup $B^{\mathrm{SU}} = B^{\mathrm{U}} \cap \mathrm{SU}_n(q)$ for $\mathrm{SU}_n(q)$. Notice that B^{SU} is the group of fixed points of the Frobenius restricted to $_n(K)$ and is also upper triangular. We have corresponding Levi decompositions $B^{\mathrm{U}} = T \ltimes U$ and $B^{\mathrm{SU}} = S \ltimes U$ where $S = T \cap \mathrm{SU}_n(q)$. By standard parabolic subgroups we mean subgroups containing B^{U} or B^{SU}. For $J \subseteq I$ let P_J^{U} or P_J^{SU} be the standard parabolic group of $\mathrm{U}_n(q)$ or $\mathrm{SU}_n(q)$ respectively corresponding to J. For fixed $J \subset I$, we have $P_J^{\mathrm{U}} = N_{\mathrm{U}_n(q)}(U_J)$ and $P_J^{\mathrm{SU}} = N_{\mathrm{SU}_n(q)}(U_J)$, both containing the same upper triangular unipotent radical, i.e. $O_p(P_J^{\mathrm{U}}) = O_p(P_J^{\mathrm{SU}}) = U_J$

The group $\mathrm{SU}_n(q)$ contains every p-subgroup of $\mathrm{U}_n(q)$ and $O_p(\mathrm{SU}_n(q)) = O_p(\mathrm{U}_n(q)) = 1$. Thus any radical p-chain of $\mathrm{SU}_n(q)$ is a radical p-chain of $\mathrm{U}_n(q)$ by Proposition 2.1. Conversely, let

$$C : U_0 < U_1 < \cdots < U_l$$

be a radical p-chain of $\mathrm{U}_n(q)$. The U_i are unipotent radicals of parabolic subgroups of $\mathrm{U}_n(q)$. Hence each U_i is conjugate to a standard unipotent radical U_{J_i}, i.e. for each i there exists $g_i \in \mathrm{U}_n(q)$ such that

$$U_i = g_i U_{J_i} {g_i}^{-1}.$$

For all $x \in F_{q^2}^*$, the matrix

$$\bar{x} = \begin{pmatrix} x & 0 & \dots & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 1 & 0 \\ 0 & & \dots & 0 & x^{-q} \end{pmatrix} \text{ stabilizes all standard } U_{J_i}.$$

In particular, this holds for $\bar{x_i}$ such that $x_i^{1-q} = (\det(g_i))^{-1}$. Moreover, $\forall g_i \in \mathrm{U}_n(q)$ there exists such an $x_i \in F_{q^2}^*$. Thus

$$U_i = g_i(\bar{x_i}U_{J_i}\bar{x_i}^{-1})g_i^{-1} = (g_i\bar{x_i})U_{J_i}(g_i\bar{x_i})^{-1}$$

where $g_i\bar{x_i} \in \mathrm{SU}_n(q)$. Thus C is $\mathrm{SU}_n(q)$-conjugate to a radical p-chain of $\mathrm{SU}_n(q)$.

For an irreducible character ρ of Z^{SU}, the center of $\mathrm{SU}_n(q)$, recall $k_d(P_J^{\mathrm{SU}}, \rho) = |\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho)|$ is the number of irreducible characters of P_J^{SU} lying over ρ (i.e. in the p-block corresponding to ρ) with q-height d. The p-part of $|\mathrm{SU}_n(q)|$ is equal to $q^{\binom{n}{2}}$. As we saw in the introduction DOC can be written:

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J^{\mathrm{SU}}, \rho) = 0, \text{ for all } \rho \text{ in } \mathrm{Irr}(Z^{\mathrm{SU}}) \text{ and nonnegative integers } d < \tbinom{n}{2}. \quad (1)$$

We now reformulate this statement using Clifford Theory. Let det be the determinant map on $\mathrm{U}_n(q)$. Then $\det(\mathrm{U}_n(q)) = \mathbb{C}_{q+1}$ and $\ker\det = \mathrm{SU}_n(q)$. Moreover, restricting the determinant map to parabolic subgroups P_J^{U} we have $\det(P_J^{\mathrm{U}}) = \mathbb{C}_{q+1}$ and $\ker\det|_{P_J^{\mathrm{U}}} = P_J^{\mathrm{SU}}$. The group P_J^{SU} is normal in P_J^{U} and hence P_J^{U} acts on the set $\mathrm{Irr}(P_J^{\mathrm{SU}})$ in the natural way. For $g \in P_J^{\mathrm{U}}$ and $\phi \in \mathrm{Irr}(P_J^{\mathrm{SU}})$, $g\cdot\phi = {}^g\phi$ where ${}^g\phi(x) = \phi(gxg^{-1})$ as defined in Section 2.4 of Chapter 2. The quotient group is cyclic

$$P_J^{\mathrm{U}}/P_J^{\mathrm{SU}} \cong \mathbb{C}_{q+1}.$$

.

Let $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$ denote the irreducible characters $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}})$ such that ϕ lies over ρ, has q-height d, and the P_J^{U} orbit of ϕ contains j characters. Then the following implies 1:

For integers $0 \le d < \binom{n}{2}$, $1 \le j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$

$$\sum_{J\subseteq I}(-1)^{|J|}|\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)| = 0. \quad (2)$$

Rather than counting characters of P_J^{SU} we count characters of P_J^{U}. Let $\chi \in \mathrm{Irr}(P_J^{\mathrm{U}})$. By Clifford Theory, χ restricted to P_J^{SU} is multiplicity free. The restrictions of two irreducible characters χ and χ' of P_J^{U} to P_J^{SU} have the same irreducible constituents or are disjoint (Lemma 2.15). If $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$, then the P_J^{U}-orbit of ϕ contains j characters. For $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$, let $\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)$ denote the subset of $\mathrm{Irr}_d(P_J^{\mathrm{U}})$ consisting of characters such that their restrictions to $\ker\det$ belong to $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$.

A character $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$ extends to $\widetilde{\phi} \in \mathrm{Irr}(T_{P_J^{\mathrm{U}}}(\phi))$. The induced character $(\widetilde{\phi}\theta)^{P_J^{\mathrm{U}}}$ is irreducible where θ is the lift to $T_{P_J^{\mathrm{U}}}(\phi)$ of an irreducible character of $T_{P_J^{\mathrm{U}}}(\phi)/P_J^{\mathrm{SU}}$. Then

$$\left|T_{P_J^{\mathrm{U}}}(\phi)/P_J^{\mathrm{SU}}\right| = \frac{q+1}{j} \text{ since } \left|P_J^{\mathrm{U}}/T_{P_J^{\mathrm{U}}}(\phi)\right| = j.$$

Let k_J be the number of P_J^{U}-orbit representatives in $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$. Then

$$\begin{aligned}\left|\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)\right| &= j \cdot k_J \\ \left|\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)\right| &= \tfrac{q+1}{j} \cdot k_J.\end{aligned}$$

Then

$$\sum_{J\subseteq I}(-1)^{|J|}|\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)| = \sum_{J\subseteq I}(-1)^{|J|} j \cdot k_J = j \cdot \sum_{J\subseteq I}(-1)^{|J|}k_J = 0$$

holds if and only if $\sum_{J\subseteq I}(-1)^{|J|}k_J = 0$ if and only if

$$0 = \tfrac{q+1}{j} \cdot \sum_{J\subseteq I}(-1)^{|J|}k_J = \sum_{J\subseteq I}(-1)^{|J|}\tfrac{q+1}{j} \cdot k_J = \sum_{J\subseteq I}(-1)^{|J|}|\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)|.$$

Hence the following equation is equivalent to 2:
For integers $0 \leq d < \binom{n}{2}$, $1 \leq j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$:

$$\sum_{J\subseteq I}(-1)^{|J|}|\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)| = 0. \tag{3}$$

We will need the case where $d = \binom{n}{2}$. Recall from Chapter 2, the isomorphism of $(F_{q^2})^*$ and $\mathrm{Irr}((F_{q^2})^*)$ so that $a_\rho \in (F_{q^2})^*$ corresponds to $\rho \in \mathrm{Irr}((F_{q^2})^*)$ under this isomorphism. This induces an isomorphism of the cyclic subgroups Z^{U} and Z^{SU} with $\mathrm{Irr}(Z^{\mathrm{U}})$ and $\mathrm{Irr}(Z^{\mathrm{SU}})$ respectively. Also recall from Chapter 2 the definition (2.4) of $\beta(\mu, a_\rho)$.

For $\rho' \in \mathrm{Irr}(Z^{\mathrm{U}})$ let $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho')$ be the set of characters $\chi \in \mathrm{Irr}(P_J^{\mathrm{U}})$ such that χ lies over ρ'. For $J \neq \emptyset$, $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho')$ consists of a unique p-block corresponding to ρ'. However if $J = \emptyset$ then $P_J^{\mathrm{U}} = \mathrm{U}_n(q)$ and $\mathrm{Irr}(\mathrm{U}_n(q), \rho')$ consists of two p-blocks one of zero defect the other of full defect. Observe $|\mathrm{Irr}_{\binom{n}{2}}(\mathrm{U}_n(q), \rho')|$ is one, the number of irreducible characters of $\mathrm{U}_n(q)$ of full q-height lying over ρ'. For $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$, let $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho)$ denote the set of irreducible characters of P_J^{U} that lie over $\rho' \in \mathrm{Irr}(Z^{\mathrm{U}})$ where ρ' lies over ρ. Then for $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$ we have the following disjoint union

$$\mathrm{Irr}(P_J^{\mathrm{U}}, \rho) = \bigsqcup_{\substack{\rho' \in \mathrm{Irr}(Z^{\mathrm{U}}) \\ \rho'|_{Z^{\mathrm{SU}}} = \rho}} \mathrm{Irr}(P_J^{\mathrm{U}}, \rho').$$

We focus now on irreducible characters in $\mathrm{Irr}(Z^{\mathrm{U}})$ and so switch the roles of ρ and ρ'. The following implies 3:

For integers $0 \leq d$, $1 \leq j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{U}})$,

$$\sum_{J\subseteq I}(-1)^{|J|}|\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)| = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and j=1;} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Given $\rho \in \mathrm{Irr}(Z^{\mathrm{U}})$, let $k_d(P_J^{\mathrm{U}}, \rho, \det, j)$ denote the number of irreducible characters $\chi \in \mathrm{Irr}(P_J^{\mathrm{U}})$ such that χ lies over ρ, has q-height d and $\chi|_{\ker \det}$ has j' irreducible constituents where j divides j'. Observe that

$$\ker(\det) = P_J^{\mathrm{U}} \cap \mathrm{SU}_n(q) = P_J^{\mathrm{SU}} \text{ as mentioned.}$$

Then

$$k_d(P_J^{\mathrm{U}}, \rho, \det, j) = \sum_{j|j'} |\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j')|.$$

We may now drop the superscript notation and restrict our attention to irreducible characters of parabolic subgroups of $\mathrm{U}_n(q)$.

Taking into account that equation (4) implies equation (3) which implies equation (2) which implies equation (1), we will have proved DOC for $\mathrm{SU}_n(q)$ if we prove the following theorem, which is the main result of this work.

Theorem 4.1 (Main) *Let $Z = Z(\mathrm{U}_n(q))$ and $\{P_J | J \subseteq I\}$ the set of standard parabolic subgroups in $\mathrm{U}_n(q)$. For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d we have*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j\text{=1;} \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove Theorem 4.1 we first break the left hand side into two sub-sums, the second of which will reduce quite spectacularly. Let us differentiate between characters χ counted by $k_d(P_J, \rho, \det, j)$ for which $\ker \chi$ contains U_J or not.

Definition 4.2 *Let $k_d^0(P_J, U_J, \rho, \det, j)$ be the number of characters counted by $k_d(P_J, \rho, \det, j)$ which contain U_J in their kernel and let $k_d^1(P_J, U_J, \rho, \det, j)$ count those characters which do not contain U_J in their kernel.*

Then

$$\begin{aligned} \sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) &= \sum_{J \subseteq I} (-1)^{|J|} (k_d^0(P_J, U_J, \rho, \det, j) + k_d^1(P_J, U_J, \rho, \det, j)) \\ &= \sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, j) + \sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) \end{aligned} \tag{5}$$

We will show the following:

Proposition 4.3 *For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \tag{6a}$$

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J,U_J,\rho,\det,j)=-\sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}}\beta(\mu,a_\rho). \tag{6b}$$

Clearly Proposition 4.3 implies Theorem 4.1.

5 Auxiliaries for the Proof of the Main Theorem

This section is dedicated to proving equation (6a), the first half of Proposition 4.3. Our first observation is that if χ is an irreducible character of P_J containing U_J in its kernel then we may consider χ as an irreducible character of the Levi subgroup L_J of P_J. We must be careful in applying the determinant map.

Let $J = \{j_1, j_2, \ldots, j_s\}$. Then L_J can be written as the following direct product:

$$L_J = \mathrm{GL}_{n_1}(q^2) \times \mathrm{GL}_{n_2}(q^2) \times \cdots \times \mathrm{GL}_{n_s}(q^2) \times \mathrm{U}_{n-2j_s}(q)$$

where $n_1 = j_1$, $n_i = j_i - j_{i-1}$ for $2 \leq i \leq s$. Then since $P_J = L_J U_J$ for $x \in P_J$, $x = lu$ where $l \in L_J$ and $u \in U_J$, so the determinant $\det(x) = \det(lu) = \det(l)\det(u) = \det(l)$ since u is unipotent.

Recall our definition

$$\mathrm{U}_n(q) = \{(a_{i,j}) \in \mathrm{GL}_n(q^2) \mid \ M = (a_{i,j})M(a_{j,i}^q)\}$$

where M is the $n \times n$ matrix with ones down the reverse diagonal.

With this definition the fixed Borel subgroup of $\mathrm{U}_n(q)$ is upper triangular. Thus for $x \in P_J$ we may write x as a block matrix:

$$x = \begin{pmatrix} A_1 & * & \ldots & & & & \ldots & * & * \\ 0 & A_2 & \ldots & & & & \ldots & * & * \\ \vdots & \vdots & \ddots & & & & & \vdots & \vdots \\ & & & A_s & & & & & \\ & & & & B & & & & \\ & & & & & \widetilde{A_s} & & & \\ \vdots & \vdots & & & & & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & & & & \ldots & \widetilde{A_2} & * \\ 0 & 0 & \ldots & & & & \ldots & 0 & \widetilde{A_1} \end{pmatrix}$$

where $A_k \in \mathrm{GL}_{n_k}(q^2)$, $B \in \mathrm{U}_{n_{s+1}}(q)$, and if $A_k = (a_{i,j})$, then $\widetilde{A_k} = M(a_{j,i}^q)^{-1}M^{-1}$.

The determinant of x as an element in P_J which is embedded in $\mathrm{U}_n(q)$ may be defined in terms of the determinant map on the component factors of the Levi subgroup L_J in P_J. We have $\det(x) = (\det(A_1)\det(A_2)\ldots\det(A_s))^{1-q}\det(B)$ since $\det(\widetilde{A_k}) = \det(A_k)^{-q}$.

Thus $k_d^0(P_J, U_J, \rho, \det, j) = k_d(L_J, \rho, \det|_{L_J}, j)$ where the determinant map $\det|_{L_J}$ is as indicated. Hence we are proving the equivalent statement:

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(L_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{7}$$

We make the following observation. Suppose $L_J = G_1 \times G_2$ where $G_1 \cong \mathrm{GL}_{n_1}(q^2)$ and $G_2 \cong \mathrm{U}_{n_2}(q)$ with $2n_1 + n_2 = n$. As a subgroup embedded in $\mathrm{U}_n(q)$ the determinant map

on elements in L_J is defined in terms of the determinant map on the factors G_1 and G_2. For $x = g_1g_2$ in L_J with $g_i \in G_i$ $\det(x) = \det(g_1)^{1-q}\det(g_2)$. Recall $\det(g_1)^{1-q}$ is denoted $\det^{1-q}(g_1)$ Then $\ker(\det) = \{(g_1, g_2) \mid \det^{1-q}(g_1)\det(g_2) = 1\}$, $\det^{1-q}(G_1) = \mathbb{C}_{q+1}$, and $\det(G_2) = \mathbb{C}_{q+1}$.

If χ_i lies over $\rho_i \in \mathrm{Irr}(\mathbb{C}_{q+1})$, then $\chi = \chi_1\chi_2$ lies over $\rho = \rho_1\rho_2$ since for $z \in Z(\mathrm{U}_n(q))$ $\chi(z) = \chi_1(1)\chi_2(1)\rho_1(z)\rho_2(z)$. Hence Lemma 2.18 implies that

$$k_d(L_J, \rho, \det, j) = \sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} k_{2d_1}(G_1, \rho_1, \det^{1-q}, j)k_{d_2}(G_2, \rho_2, \det, j). \tag{8}$$

We proceed by induction on n or equivalently by induction on m, where $n = 2m$ or $2m+1$.

5.1 Small Case

Let $n = 1$ so that $m = 0$ and I is empty. Then we have but one Levi subgroup, $\mathrm{U}_1(q)$ itself which is equal to its center. The determinant map is just the identity and hence $\ker\det$ is trivial so that the left hand side of equation (7) is

$$k_d(\mathrm{U}_1(q), \rho, \det, j) = \begin{cases} 1, & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Certainly this is equal to the right hand side of equation (7) since we have but one partition of 1 and $\beta((1), a_\rho) = 1$
Let $m = 1$ so that $n = 2$ or $n = 3$. In either case we have but two Levi subgroups $\mathrm{U}_n(q)$ and the Borel Levi subgroup L_I. First suppose that $n = 2$. Then $L_I = \mathrm{GL}_1(q^2)$ and we may apply 2.23 and 2.22 directly. The left hand side of 7 is

$$k_d(\mathrm{U}_2(q), \rho, \det, j) - k_d(\mathrm{GL}_1(q^2), \rho, \det^{1-q}, j) = \begin{cases} \beta((2), a_\rho) - 0, & \text{if } d = 1 \text{ and } j = 1; \\ \beta((1^2), a_\rho) - 0, & \text{if } d = 0 \text{ and } j = 2; \\ q\beta(1^2, a_\rho) - \overline{\beta}(1, a_\rho), & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This is equal to the right hand side of equation (7).

We continue to assume that $m = 1$. Now suppose that $n = 3$. Then $L_I = \mathrm{GL}_1(q^2) \times \mathrm{U}_1(q)$ so that 8 implies

$$k_d(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q), \rho, \det, j) = \sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} k_{2d_1}(\mathrm{GL}_1(q^2), \rho_1, \det^{1-q}, j)k_{d_2}(\mathrm{U}_1(q), \rho_2, \det, j)$$

which is nonzero only for $d = 0$. Then the left hand side of equation (7) is

$$k_d(\mathrm{U}_3(q), \rho, \det, j) - k_d(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q), \rho, \det, j)$$
$$= \begin{cases} \beta((3), a_\rho) - 0, & \text{if } d = 3 \text{ and } j = 1; \\ \beta((2,1), a_\rho) - 0, & \text{if } d = 1 \text{ and } j = 1; \\ \beta(1^3, a_\rho) - 0, & \text{if } d = 0 \text{ and } j = 3; \\ q^2\beta(1^3, a_\rho) - (q-1)(q+1), & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This is equal to the right hand side of equation (7).

5.2 Inductive Case

We assume that equation (7) holds for all dimensions strictly less than n. Our first observation is that for fixed J with minimal element $j_1 = k$ we may write $L_J = \mathrm{GL}_k(q^2) \times L_{J'}$ where $L_{J'}$ is a levi subgroup in $U_{n-2k}(q)$ and $J' = \{j_i - k \mid 2 \leq i \leq s\}$. Note $|J'| = |J| - 1$. We will use superscripts to indicate the dimension of the ambient group when necessary. So for example $L_{J'}$ will be written as $L_{J'}^{n-2k}$. For such a J we have

$$k_d(\mathrm{GL}_k(q^2) \times L_{J'}^{n-2k}, \rho, \det, j) = \sum_{\substack{2d_1 + d_2 = d \\ \rho_1 \rho_2 = \rho}} k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \det{}^{1-q}, j) k_{d_2}(L_{J'}^{n-2k}, \rho_2, \det, j).$$

We remark that $k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \det{}^{1-q}, j) = 0$ for k not divisible by j. We could eliminate from our sum all J whose smallest members are not all multiples of j. This isn't necessary though since the contribution is just zero. In fact we may discard all Js not contained in $\{j, 2j, 3j, \ldots\} \subseteq I$ but again this isn't necessary for our induction.

Definition 5.1 *Fix $k \geq 1$ and let $\mathbf{J}_k$ be the collection of all $J \subseteq I$ with minimal member k.*

We have

$$\begin{aligned}
\sum_{J\in\mathbf{J}_k}(-1)^{|J|}k_d(L_J,\rho,\det,j) &= \sum_{J'\subseteq I^{m-k}}(-1)^{|J|}k_d(\mathrm{GL}_k(q^2)\times L_{J'}^{n-2k},\rho,\det,j)\\
&= \sum_{J'\subseteq I^{m-k}}\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}(-1)^{|J'|+1}k_{2d_1}(\mathrm{GL}_k(q^2),\rho_1,\det{}^{1-q},j)k_{d_2}(L_{J'}^{n-2k},\rho_2,\det,j)\\
&= -\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}\left(k_{2d_1}(\mathrm{GL}_k(q^2),\rho_1,\overset{1-q}{\det},j)\sum_{J\subseteq I^{m-k}}(-1)^{|J|}k_{d_2}(L_J^{n-2k},\rho_2,\det,j)\right)\\
&= -\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}\left(\sum_{\substack{\mu_1\vdash k\\ n'(\mu_1)=d_1\\ j|\gcd(q+1,\lambda(\mu_1))}}q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\times\sum_{\substack{\mu_2\vdash(n-2k)\\ n'(\mu_2)=d_2\\ j|\gcd(q+1,\lambda(\mu_2))}}\beta(\mu_2,a_{\rho_2})\right)\\
&= -\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu_1\vdash k\\ \mu_2\vdash(n-2k)\\ n'(\mu_1)=d_1\\ n'(\mu_2)=d_2\\ j|\gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}}q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})
\end{aligned}\tag{9}$$

by our inductive assumption, for indeed $n-2k$ is strictly less than n.

Summing over all possible values for k and switching the order of summation we have

$$\begin{aligned}
&\sum_{k=1}^{m}\sum_{J\in\mathbf{J}_k}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&\quad= -\sum_{k=1}^{m}\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu_1\vdash k\\ \mu_2\vdash(n-2k)\\ n'(\mu_1)=d_1\\ n'(\mu_2)=d_2\\ j|\gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}}q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})\\
&\quad= -\sum_{\substack{\mu\vdash n\\ n'(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ |\mu_1|\neq 0\\ j|\gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}}\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}}q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})\\
&\quad= -\sum_{\substack{\mu\vdash n\\ n'(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ |\mu_1|\neq 0\\ j|gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}}q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}\beta(\mu,a_\rho).
\end{aligned}\tag{10}$$

Now we remind the reader that from Corollary 2.11 we have the following for each $\mu \vdash n$

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = q^{l(\mu)/j-\delta(\mu)}. \tag{11}$$

Notice that this sum includes the pair $(\mu_1,\mu_2) = (\emptyset,\mu)$ and that for this particular pair

$$q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = 1.$$

We are now ready to prove equation (7):

$$\begin{aligned}
\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J,\rho,\det,j) &= k_d(U_n(q),\rho,\det,j) + \sum_{\emptyset\neq J\subseteq I}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&= k_d(\mathrm{U}_n(q),\rho,\det,j) + \sum_{k=1}^{m}\sum_{J\in\mathbf{J}_k}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu,a_\rho)\\
&\qquad -\left(\sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu,a_\rho) - \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho)\right)\\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho).
\end{aligned} \tag{12}$$

And we are done.

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